**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part II. Optimization methods

We considered different problems of minimization integral functionals. The dependence of these functionals was being direct here. However for the practical problems this dependence can be non-direct. The minimizing functional depends from the state functions; end the dependence of the state functions from unknown values (controls) is described by the state equations. These problems are called optimization control problems. We will consider some methods of its solving.

## Lecture 10. Easiest optimization control problems

We consider at first the maximization of the flight of the missile as an example of the optimization control problem. Then we will give the general problem statement for the case with one state function and one control. We will solve it with using the Pontryagin’s maximum principle. It is a necessary condition of optimality. We will give an example of its analysis. Then we will consider the vector case of the optimization control problems.

### 10.1. Maximization of the flight of the missile

We considered in the Introduction the problem of the maximization of the missile flight. This is the flight on the vertical plane. The missile moves by driving force during the time interval . Then it moves by gravitation only. Using Newton low, we had the equations of the movement



where *m* is the mass, *Р* is the weight, *Fx* and *Fy* are horizontal and vertical driving forces. We have the following formulas for determination of the forces.

*Fx* = *F* cos *u*, *Fy* = *F* sin *u*, *P* = *mg*,

where *F* is the known driving force, *g* is the gravitational acceleration, *u* is angle. Then we have the state equations

  (10.1)

where the acceleration *а = F/m* is known; and the function *u* is the control (we can choose it).

We have also the initial conditions

  (10.2)

So the missile is in the origin of the coordinate at the initial time; and initial velocity is zero.

We proved before, that the length of the flight is calculated by the formula

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where *T* is the time of the finish of the fuel. We have the following extremum problem.

**Problem 10.1**. *Find the function u = u*(*t*), *which maximize the value L*.

This is an example of the *optimization control problem*. The maximizing value *L* depends from the unknown function *u* (the control) non-directly in this case. It depends from state functions *x* and *y*; and the state functions depends from the control by state equations (10.1), (10.2). Now we will consider the general form of the optimization control problems.

### 10.2. Problem statement

Consider the control system described by the equation

  (10.3)

with the initial conditions

 *х*(0) = *х*0,(10.4)

where *x = x*(*t*) is the state function, *u = u*(*t*) is the control, *f* is the given function, *х*0 is the given initial state. For all control *u* we can determine the solution *x* of the problem (10.3), (10.4).

As a rule the control *u* is chosen from a set of the admissible controls *U*. Suppose this set is determined by the formula

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where *a*, *b* are given functions. Consider the value

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where *g* and *h* are given functions.

**Problem 10.2**. *Find the function u = u*(*t*) *from the set U*, *which minimize the value I, where the relation between the state and the control is described by the equalities* (10.3), (10.4).

### 10.3. Maximum principle

Using Lagrange multipliers method determine Lagrange function



where the function *p* is Lagrange multiplier. If the function *x* is the solution of the equation (10.3), then the value *L* is equal to *I.* Determine the function

 *H*(*t*,*u*,*x*,*р*) = *р f*(*t*,*u*,*x*) – *g*(*t*,*u*,*x*). (10.5)

Then we have the equality

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Suppose *u* is a solution of our problem (the optimal control). So we have then inequality

Δ*I* = *I*(*v*,*y*) – *I*(*u*,*x*) ≥ 0 ∀*v*∈*U*,

where *х* and *у* are the solutions of the problem (10.3), (10.4) for the controls *u* and *v.* Then previous inequality can be transformed to

 Δ*L* = *L*(*v*,*y*,*р*) – *L*(*u*,*x*,*р*) ≥ 0 ∀*v*∈*U*, ∀*р* (10.6)

Find the value

,

where

Δ*H* = *H*(*t*,*v*,*y*,*р*) – *H*(*t*,*u*,*x*,*р*), Δ*h* = *h*[*y*(*T*)] – *h*[*x*(*T*)].

Suppose the functions of the problem statement are smooth enough. Using Taylor formula we get

*h*[*y*(*T*)] = *h*[*x*(*T*) + Δ*x*(*T*)] = *h*[*x*(*T*)] + *hх*[*x*(*T*)] Δ*x*(*T*) + *η*1,

where , *hх* = *dh*/*dx*, and *η*1 is the high term with respect Δ*х*(*T*). Then we have

*H*(*t*,*v*,*y*,*р*) = *H*(*t*,*v*,*х*+Δ*x*,*р*) = *H*(*t*,*v*,*х*,*р*) + *Hх*(*t*,*v*,*х*,*р*) Δ*x* + *η*2 =

= *H*(*t*,*v*,*х*,*р*) + *Hх*(*t*,*u*,*х*,*р*) Δ*x* + *η*2 + *η*3,

where *Hх* = ∂*H*/∂*x*, and *η*2 is the high term with respect Δ*х*,

*η*3 = [*Hх*(*t*,*v*,*х*,*р*) – *Hх*(*t*,*u*,*х*,*р*)] Δ*x*.

So the inequality (10.6) can be transformed to



where

Δ*uH* = *H*(*t*,*v*,*x*,*р*) – *H*(*t*,*u*,*x*,*р*),



Using the integration by parts we get



because of the equalities *х*(0) = *у*(0) = *х*0. So we obtain the inequality

  (10.7)

Using the arbitrariness of the function *p* choose it such that this inequality transforms to the easy one. Suppose *p* is the solution of the equation

  (10.8)

with final condition

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The system (10.8), (10.9) is called the *adjoint system*. Then the inequality (10.7) is transformed to

  (10.10)

The left side of this inequality is the sum of two values. If the control *v* is close enough to the optimal control *v*, then the high term *η* has the high order of the infinitesimality. Then the sign of the sign is determine by the first term. So we obtain the inequality



We can prove that the sign of the integral is determine by the sign of the value under the integral. So we get the inequality



It can be transform to

  (10.11)

This equality is called the *maximum principle*.

**Theorem 10.1.** *The solution of the Problem* 10.2 *satisfies the maximum principle.*

Thus, we have the system that includes the state system, the adjoint system and the maximum principle with respect to the unknown functions *x*, *p*, *u.*

### 10.4. Example

Consider the system

  (10.12)

We have the set of the admissible control

*U =* {*u* | | *u*(*t*) | ≤ 1, *t*∈(0,1)}

and the integral



We have the problem of its minimization on the set *U*.

Determine

*f*(*t*,*u*,*x*) = *u*, *x*0 = 0, *T*=1, *g*(*t*,*u*,*x*) = (*u*2 + *x*2)/2 , *ϕ**a* = -1, *b*= 1.

Find the function *H* from the equality (10.5)

*H = H**u*) *=**р u* – (*u*2 + *x*2)/2 .

The adjoint system (10.8), (10.9) has the form

  (10.13)

The maximum principle (10.11) is transformed to

  (10.14)

Hence we have the system (10.12) – (10.14) for finding three unknown functions *u*, *х*, *p*. Find the solution of the maximum principle. Using stationary condition we get

∂*H*/∂*u* *=**p* – *u =*.

It has the unique solution *u* = *p*. The second derivative of *H* is negative. So we have the point of the maximum in really. However we do not know if it satisfies the given constraint. The value
*u* = *p*can be different position with respect to the interval [-1,1] (see Figure 10.1). If *p*1 the function *Н* decries on this interval. So it maximum is reached on the minimal point; then
*u*=-1If *p*1 then *Н* increases on this interval. So it has the maximum on the maximal admissible value, and we find *u* = 1If |*p*≤1the value *u* = *p* is admissible; so it can be the solution of the maximum principle. Note that *p* depends from the time. So our result depends from the time too. Therefore we get the formula

 . (10.15)



Figure 10.1. Conditional maximum of the function *H.*



Figure 10.2. Solving of the maximum principle if the function *p* is given.

The formula (10.15) give the solution of the maximum principle (10.14) if we know the function *p* (see Figure 10.2). So we obtain the system (10.12), (10.13), (10.15) with respect to three unknown functions *u*, *х*, *p*. Our final problem is solving of this system. It can be solved approximately.

Let the control *uk* on the iteration *k* is known. Then we can find the corresponding state function *хk* from the system

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Then we solve the adjoint system

 

The control *uk*+1 on the next iteration determines by the formula

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We now try to apply the method of successive approximations directly to the system of optimality conditions. Let *u*0 be an initial approximation of the control. It must belong to the set of admissible controls, i. e.,

-1 ≤ *u*0(*t*) ≤ 1 , *t*∈[-1,1] .

Integrating this inequality from aero to an arbitrary *t* and using the state equation, we have



where *x*0 is the initial approximation of the state.

Further integration of the obtained relation from *t* to unity (the terminal point) yields



where *p*0 is the initial approximation of the adjoint state *p*.Since all values of *p*0belong to the segment [-1,1], following (10.15), we find the next approximation of the control *u*1(*t*) *= p*0(*t*)*.* We have

-1/2 ≤ *u*1(*t*) ≤ 1/2 , *t*∈[-1,1] .

Integrating this inequality, we get



Further integration of the obtained inequality from *t* to unity gives



Then (10.15) implies that the next approximation of the control satisfies the inequality

-1/4 ≤ *u*2(*t*) ≤ 1/4 , *t*∈[-1,1] .

Repeating the above calculations for the next iteration, we obtain

-1/8 ≤ *u*3(*t*) ≤ 1/8 , *t*∈[-1,1] .

In the general case, for the *k-*thiteration we have the following estimate:



Thus, we have established the convergence  as .

The obtained results show that the sequence constructed using the method of successive approximations converges to the function *u*\*, which is identically equal to zero, for any initial approximation of the control chosen from *U.*

A natural question arises of *whether u\* will be a solution of the consid­ered optimal control problem?* To answer this question, we return to the formulation of the problem. Since the integrand in the functional to be minimized is nonnegative, we have *I* ≥ 0 for every admissible control. Zero value of the functional is achieved if and only if

*u*(*t*) = 0, *х*(*t*) = 0, *t*∈[-1,1].

The zero control is admissible. Moreover, according to problem (10.12), it defines the state function, which is also identical zero. Thus, it is the admis­sible control *u\** that makes the functional vanish; furthermore, the functional does not assume negative values. Therefore, this optimal control problem has a unique solution, which is the one we found as a result of approximate solution of the optimality conditions.

**Outcome**

* Optimization control problems consist of a state equation, a set of admissible controls and a minimizing functional.
* Optimization control problems can be analyzed with using Lagrange multipliers methods.
* Necessary conditions of the optimality for the optimization control problem consist of the state system, the adjoint system, and Pontryagin’s maximum principle.
* The adjoint system consists of the adjoint equation and the final condition.
* Maximum principle is a problem of the maximization of some function *H*.
* Necessary conditions of the optimality for the optimization control problem can be solved by iterative method.

### Task 10. The easiest optimization control problem

The state of the system is described by Cauchy problem



We have the problem of the minimization of the functional



on the set



 Table 10.1. The values of the parameters.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Variant |  | *T* | *x*0 |  |  | *a* | *b* |
| 1 |  | 1 | 0 | 0 |  | -1 | 1 |
| 2 |  | 2 | 1 |  | 0 | 0 | 1 |
| 3 |  | 3 | -1 |  |  | 0 | 2 |
| 4 |  | 1 | 2 |  |  | -1 | 1 |
| 5 |  | 2 | 0 | 0 | *x* | -2 | 2 |
| 6 |  | 3 | 1 |  | 0 | -1 | 0 |
| 7 |  | 1 | -1 |  |  | -1 | 1 |
| 8 |  | 2 | -2 |  |  | 0 | 1 |
| 9 |  | 3 | 1 | 0 |  | 1 | 2 |
| 10 |  | 1 | -1 |  |  | 0 | 1 |

Steps of the task:

1. Denote the concrete problem statement.
2. Determine the function *H*.
3. Determine the adjoint equation with the final condition.
4. Find the control from the maximum principle.
5. Determine the iterative method for solving the necessary conditions of optimality.

### Next step

We considered the optimization control problems with unique state function and the unique control. Then we will try to extend these results to the case with many state functions and controls.